17. The above formulas express the energy in absolute units; a horsepower contains 736×10^7 of these units. Therefore

$$P = \frac{mA}{736 \times 10^7} \tag{3}$$

where P is the developed energy expressed in horsepowers. In the atmosphere and the ocean problems it is more convenient to employ tons instead of grams; therefore we write

$$P = \frac{NA}{7360} \text{HP}. \tag{4}$$

where N is the mass of moving air or water expressed in tons per second of the C. G. S. system. A is as before the number of solenoids in the inclined solenoidal surface or streak.

18. The circulation represented in figure 2 can also be conceived at two different currents of water; the one current consisting of cold, specifically heavier water that flows down from the cold source toward the warm source; the other current of warm, lighter water that rises from the source of heat and flows toward the source of cold. There are, in fact, two actual waterfalls, the one of denser water that sinks and the other of lighter water that rises. Both these falls develop kinetic energy in the same way as do the waterfalls of our rivers. The only difference is that, instead of the total specific gravity of the water of the river computation, we here have to employ the difference in specific gravity between the flowing and the surrounding, adjacent water. This is a natural consequence of the Archimedean principle of pressure of adjacent surrounding water against the flowing water. Employing the thus reduced specific gravity, the actual height of the waterfall, and the stream discharge, the computation of the energy gives the same result as does the solenoidal formula.

19. One can also compute the correct amount of developed energy by employing the actual specific gravity and the height of the waterfall if a corresponding reduction is applied to the mass of the flowing water; or we may use the actual specific gravity and the stream discharge in combination with an appropriately reduced height of waterfall. I have found the last of these procedures the most practical because that method permits the direct substitution of tons of flowing water or air for the cubic meters of water of the river they are compared with.

20. In the atmosphere, the snow and ice covered mountain tops and high plateaus correspond with the cold source C of figure 2, and the warm surface water of the warm ocean with its warm currents correspond with the warm source W. The circulation of the air that exists in the atmosphere is of the same kind and nature as the circulation of the water shown in figure 2. Thus the air warmed above the warm surface of the ocean rises and spreads out horizontally until it comes in contact with the cold mountain tops. Here it cools, sinks, and returns along the earth's surface to the warm ocean only to repeat the circulatory process. The warm current of air above and the cold current below are separated by an inclined surface corresponding to the inclined streak of figure 2. This atmospheric surface contains the solenoids that induce and maintain the atmospheric circulation. The amount of kinetic energy developed by the two air currents can be estimated according to the manner above described, either employing formula (4) or by comparing the currents with the waterfalls.

21. The circulation of the water in the atmosphere is also like the scheme of figure 2. In this case the source of heat is the warm ocean surface whose water particles evaporate and rise in a gaseous form into the atmosphere; the cold source, C, is the point in the atmosphere where the water vapor is cooled and condensed into rain or snow. These forms fall and form rivers that flow back to the ocean, i. e., back to the source of heat W. In this circulation an immense amount of kinetic energy is produced—of it we use a very small fraction in our hydroelectric and water-power plants. The largest part of the energy of this circulation is consumed in producing the wind, as the Swedish oceanographer, Prof. Otto Peterson, has shown. The computation of this energy by the above methods offers no special difficulties.

22. The ocean currents also appear to follow the scheme of figure 2. Let us consider, for instance, the Gulf Stream. It has its origin in the great sargasso vortex which carries the sun-warmed ocean water of the Tropics downward to a depth of 600 meters. This is the source of heat W, of the Gulf Stream. From here the warm Gulf Stream water flows along the Atlantic trough northward until it reaches the ice of the Arctic Ocean. This ice corresponds to the cold source C of figure 2; it cools the Gulf Stream water which sinks to the depths along which it flows back as a cold undercurrent toward W in the Tropics. Here it is again warmed, rises to the surface, and again wanders northward. On its northward course the upper portion of the Gulf Stream becomes shallower; under the Tropics it is 600 meters deep but at Spitzbergen it is only 200 meters deep. The surface dividing the warm upper stream from the cold undercurrent is therefore inclined like the sloping streak of figure 2; therefore, this streak contains a number of solenoids, amounting to about 150,000, according to hydrographic observations. The mass of water that flows in the Gulf Stream is estimated at 25,000,000 cu. m. per second. Therefore, and by equation (4), the Gulf Stream delivers about 500,000,000 HP. This amount of energy is, of course, applied to the task of driving the Gulf Stream itself, whereby the internal friction of the water reconverts it into heat. The Gulf Stream may be compared to a river that discharges 25,000,000 cu. m. per second over a waterfall $1\frac{1}{2}$ meters high. Such a waterfall would develop the same amount of energy as does the Gulf Stream.

23. In order to be able to make such numerical estimates of the energy of the atmospheric currents we must have the proper data at appropriately located mountain stations and kite stations.

24. For the present we see from the foregoing that the simple experiment presented in figure 2 possesses many large and important counterparts in the atmosphere and the hydrosphere. Indeed, it can hardly be otherwise since it is itself a picture in miniature of the powerful heat engine that creates the currents of the wind and the ocean.

SOME RECENT RESEARCHES ON THE MOTION OF FLUIDS.

By HARRY BATEMAN, M. A., Ph. D.

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1. The early attempts of mathematicians to calculate the distribution of velocity in a fluid containing a solid body either at rest or in motion, led to conclusions which do not agree with experimental results.

In the continuous potential flow of a perfect fluid it was found, for instance, that a fluid of infinite extent offers no resistance to uniform motion of the body, provided the motion of the fluid is steady. This is the so-called *paradox of d'Alembert* (1). The result is contradicted by experience, although it should be remembered that for a spindle-shaped body of stream-line form the resistance is very small (2) and in actual experiments the fluid either has a free surface or is inclosed in a vessel of finite size.

The assumption that the velocity is everywhere continuous also led to the conclusion that fluid issuing from the mouth of a tube immediately spreads out in all directions (3), whereas in reality the fluid forms, at all events for some distance, a more or less compact stream. In 1847 Stokes (4), while discussing the motion of a fluid contained within a rotating prism whose cross section was a sector of a circle, came to the conclusion that when the angle of the sector was greater than two right angles a surface of discontinuity would form if the fluid were perfect, but that there would be no true surface of discontinuity in the case of a viscous fluid.

In 1868 Helmholtz (5) pointed out that whenever the velocity in the continuous potential flow exceeds a certain limit, the pressure becomes negative and the liquid tears asunder forming either a cavity or a surface of discontinuity. There is no doubt that a kind of cavitation actually exists in certain motions of real fluids, being assisted in the case of water by the air which is dissolved in it. The idea was developed by Lord Kelvin (6) in a remarkable paper "On the formation of coreless vortices by the motion of a solid in an inviscid incompressible fluid," in which he concludes that if the square of the velocity of a spherical solid exceeds $\frac{8}{5} P$, where P is the pressure in the undisturbed fluid at infinity, cavitation will commence at the back of the sphere and coreless vortices will be periodically formed and shed off behind the sphere during its motion through the fluid. This result is of interest in connection with the recent developments which will be described in §2; its importance has recently been emphasized by J. B. Henderson (7). D'Alembert's paradox is considered by some writers (8) to indicate that a surface of discontinuity must form when a solid body moves through a perfect fluid. Duhem (9) on the other hand regards it as implying the impossibility of a permanent régime and has shown that the paradox still holds when there are surfaces at which the velocity is discontinuous, provided the surfaces do not extend to infinity or, in the alternative case, provided the discontinuity vanishes at infinity at least as rapidly as the velocity of the fluid itself and in such a manner that a certain integral over a large surface inclosing the fluid, vanishes when this surface recedes to infinity. Villat (10) maintains, however, that a surface of discontinuity which extends to infinity, can exist when there is a permanent régime, but that the discontinuity of velocity does not satisfy the conditions laid down in Duhem's theorem, as is indicated by the mathematical analysis in a particular example. Consequently the possibility of a surface of discontinuity behind a moving body is not excluded by Duhem's argument. Moreover, M. Brillouin (11) has shown that if the pressure vanishes at infinity and there is a permanent régime, when a solid body moves uniformly through a perfect fluid there must be points at which the pressure is negative unless there is at least one surface of discontinuity which extends to infinity. This is a generalization of the result obtained by Lord Kelvin.

The mathematical theory of the motion of a perfect fluid in which there are vortex sheets or surfaces of discontinuity at which one portion of fluid glides past another, was first definitely applied to practical problems by Helmholtz (12). In the first instance the surface of discontinuity was introduced simply as a free surface of the stream of fluid flowing from a large reservoir into a narrow channel. Helmholtz concluded that the ultimate width of the stream would be half that of the channel, a result which is not very far from the truth (13). He was thus able to give a fairly satisfactory mathematical theory of jets, which accounted for the well-known instability of gaseous jets (14). In the case of a jet of fluid in air a finite discontinuity in velocity is inadmissible owing to viscosity, but it is conceivable that Helmholtz's theory may be arrived at in the limiting case when the viscosity tends to zero.

The theory of surfaces of discontinuity was afterwards extended to the case in which a solid moves through a fluid, and the mathematical analysis was developed by Kirchhoff (15), Rayleigh (16), and many other writers (17). Considerable progress has been made recently in this theory of discontinuous potential flow, by Levi-Civita (18), Cisotti (19), and Villat (20).

The theory has been used to determine the resistance met by a solid body moving through a perfect fluid, on the assumption that the wake behind the body is a region of constant (or hydrostatic) pressure bounded by a surface of discontinuity extending to infinity. A definite finite value is found for the resistance, and so the theory is not ruled out on account of D'Alembert's paradox. The theory agrees with experiment inasmuch as the resistance is found to be proportional to the square of the velocity of the body, but the calculated value for the resistance in the case of a plane lamina moving through air differs from the experimental value (21).

The theory of discontinuous motion has been attacked by Lord Kelvin (22), who claims that such a motion is inconsistent with his theorem of least energy; that a surface of discontinuity is unstable (23) and would also disappear on account of viscosity. Another serious objection is that the mass of "dead water" which is supposed to be carried along behind a body moving through a fluid, would have an infinite kinetic energy and this would imply that an infinite amount of kinetic energy is given to the fluid by the motion of the body. Since, however, the velocity of the body is supposed to be maintained by some agency, it has been thought that the type of motion in question might (conceivably) be approximated to asymptotically as time elapses, though it could not be established in a finite time (24).

The whole matter has been reviewed at some length by Lanchester (25) who points out that Lord Kelvin's minimum theorem involved the hypothesis of continuity, and so the first objection can be set aside. In connection with the other objections Lanchester states his views as follows:

(1) That whatever may be the value of the viscosity, the *initial* motion from rest obeys the Eulerian equations, i. e., the motion is continuous (26).

(2) That the discontinuous system may, in a viscous fluid, be regarded as arising by evolution from a motion initially obeying the mathematical equations of continuous motion.

(3) That in fluids possessing different values of kinematic viscosity the time taken for the evolution of the discontinuous system is greater when the kinematic viscosity is less, and vice versa.

(4) That the ultimate development of the discontinuous system of flow is more complete the less the value of the kinematic viscosity, and vice versa.

The possibility of the solution of the equations of motion of a viscous fluid becoming discontinuous when the viscosity approaches the value zero, may perhaps be illustrated by a consideration of the equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}.$$

This equation possesses a solution of the form

$$u=F(x+Ut),$$

if

$$U\frac{dF}{dx}+F\frac{dF}{dx}=\nu\frac{d^2F}{dx^2}$$

or

$$(F+U)^2\pm a^2=2\nu\frac{dF}{dx}$$

where a is a constant. The solution is thus either

$$u+U=a\tan\frac{(x+Ut-c)a}{2\nu}$$

or

$$\frac{a-u-U}{a+u+U}=e^{\frac{a}{\nu}(x+Ut-c)},$$

according as the + or − sign is taken. In the first case there is no definite value of u when ν tends to zero, while in the second case the limiting value of u is either $a-U$ or $-a-U$ according as $x+Ut$ is less or greater than c. The limiting form of the solution is thus discontinuous.

It is clear from this example that the question of the limiting form of the motion of a viscous fluid when the viscosity tends to zero requires very careful investigation. So far very little has been done on these lines, but an approximate mathematical theory of the motion of a fluid whose viscosity is very small has been proposed by Prandtl (27) and developed by some of his pupils (28). The chief characteristic of the theory is the assumption that the motion differs very little from a continuous potential flow outside a "thin layer of transition" and that within this layer there is a rapid fall of velocity; for the motion in this layer the equations of motion of a viscous fluid are used with a few simplifications.

The conclusions to which Prandtl comes are very similar to those in Lanchester's treatise. In the case of the flow of fluid round a plate, when the flow is directed at right angles to the plate at an infinite distance from it, the motion differs very little from the continuous potential flow at the very beginning of the motion, being changed very little by the thin layer of transition covering the edges. Soon, however, the fluid separates from the plate and in consequence of the friction at the wall a stream of fluid containing vortices issues from the layer of transition. The type of flow now changes behind the place of separation and a kind of vortex sheet or surface of discontinuity appears to form, only to be broken up on account of its instability into separate vortices. These conclusions have been verified by a series of experiments. A photographic reproduction of the pictures representing the flow is given in Prandtl's paper.

A complete mathematical study of the motion of a viscous fluid as its velocity increases is very desirable. The motion of a sphere in a viscous fluid when the velocity of the sphere is small has been studied very thoroughly by C. W. Oseen (29); he has made some allowance for the "inertia terms," i. e., the terms in the equations of motion which involve products of the component velocities and their derivatives, and he finds that when $\rho aU/\mu$ is small —ρ being the density, μ the coefficient of viscosity of the fluid, a the radius of the sphere, and U its velocity —the resistance is given to a second approximation by the formula:

$$R=6\pi a\mu U\left[1+\tfrac{3}{8}\frac{\rho aU}{\mu}\right].$$

A similar result has been obtained recently by R. W. Burgess in a paper which has just been presented to the American Journal of Mathematics. Burgess has, moreover, removed a defect in Oseen's theory and has obtained the modified stream-line function by a simple process.

The first approximation for R, of course, agrees with Stokes's well-known formula. An approximate formula for the resistance to the uniform motion of a right circular cylinder has been obtained by Lamb (30), the method of derivation being analogous to that used by Oseen.

In these investigations it is assumed that the motion is steady. There is, of course, vortex motion which is appreciable only in the wake, but there are no isolated vortices or cavities and no surfaces of discontinuity. If a steady motion exists, the origin of these other types of motion must be attributed to chance disturbances and a possible instability of the steady state. Since, however, in actual experiments a finite velocity of a moving solid is attained gradually, the turbulent or discontinuous motion may begin when the velocity exceeds a certain limit, depending on the viscosity, as in the theory of Osborne Reynolds (31). In this theory it is recognized that a possible criterion of stability of a given state of motion can depend only on the ratio $\frac{\rho aU}{\mu}$, where a is a characteristic length and U a characteristic velocity associated with the motion. Osborne Reynolds was led to the idea that turbulence sets in when this quantity exceeds a certain limit. This theory has been discussed with conflicting conclusions by Lord Kelvin (32), Lord Rayleigh (33), H. A. Lorentz (34), W. McFadden Orr (35), F. R. Sharpe (36), V. W. Ekman (37), C. W. Oseen (38), A. Sommerfeld (39), G. Hamel (40), R. von Mises (41), and other writers. The question must still be regarded as unsettled.

On account of instability it is difficult to understand how a surface of discontinuity could be approximated to during the course of the motion of a viscous fluid; nevertheless the results which are obtained by means of the theory apparently agree qualitatively with experiments, so that it would be unwise to reject the theory simply on account of this difficulty.

One serious objection which can be urged against the theory on experimental grounds, is that the theory does not account for the variation of pressure observed over the back of a square plate moving through air.

The distribution of pressure over both faces of a plane lamina moving with constant velocity through air has been determined experimentally by several observers (42). In some cases a whirling table was used, but the results obtained in this way are not satisfactory. The best observations have been made by carrying the lamina in a moving vehicle as in some of Langley's experiments. Armand de Gramont, Duc de Guiche, has recently adopted this method in an elaborate series of experiments carried out in a motor car and has made a number of beautiful diagrams which show very clearly that the pressure on the back of a thin square lamina is less than the atmospheric pressure over an area bounded by the leading edge and a curve which recedes toward the rear edge of the plate as the angle of inclination increases from 0° up to a critical angle of about 20°. When this angle is exceeded the pressure on the back of the plate is everywhere less than the atmospheric.

The critical angle differs from that found by Eiffel in his experiments with a stationary plane in a current of air and this indicates that results obtained by one method of experiment can not be applied with full rigor to cases which would correspond to the other method of experiment.

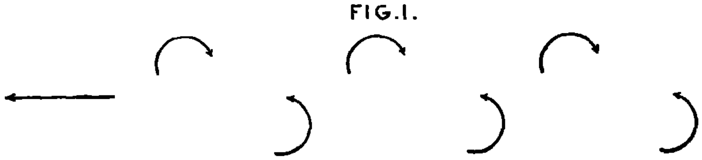
Unless the presence of the motor car alters the flow of air the exact cause of the difference between the two cases is not easily detected. In the mathematical theory it has generally been assumed that the flow of fluid around a stationary obstacle can be deduced at once from the corresponding flow, in which the fluid is at rest at infinity and the body moves through it, by simply impressing on the fluid and the body a velocity which will annul the velocity of the body. This may, however, only be true when the motion is steady. The question has been raised again and discussed in a recent paper by J. B. Henderson (43) who refers to some experiments made by Dubuat in 1786. This experimenter measured the force required to tow a plate in still water, and also the force required to hold the same plate stationary in a stream, the relative motion of plate and water being the same in the two cases. He found the ratio between the two forces to be 1.3:1.

A possible explanation of the difference, assuming it to be real, is that the eddy phenomena are not the same in both cases. Eddies arise from instability in the steady motion, and to prove the equivalence it would have to be proved, not only that the inception of instability, but also that the resulting motion following on instability depend solely on the relative motion (43).

It is very probable that conditions in which the motion of the fluid is not really steady play an important part in experimental work and this brings us to the consideration of motions which are permanent because they are periodic.

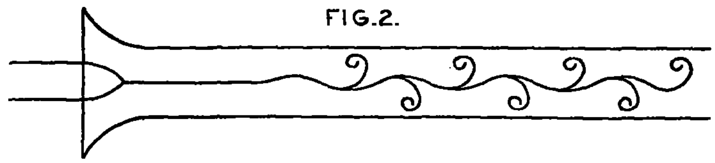
2. In 1908 H. Bénard (44) discovered that when the surface of a liquid is parted by a thin vertical prism which is moved with uniform velocity parallel to its plane of symmetry, two parallel sets of gyration centers form behind the prism. The vortices belonging to a row are at equal distances apart and have the same sense of rotation which is opposite, however, to that of the vortices in the other row. A central dissymmetrical space was detected at the back of the prism and was identified with the vibration zone where the alternate vortices are formed. At two instants separated by half a period, the appearance of the zone is exactly symmetrical with regard to the plane of symmetry of the obstacle. At first the vortices have the same velocity as the moving prism, but quickly slacken, at the same time diverging to the right and left. They quickly attain their transversal limit, longitudinal equidistance, and limiting speed, which are preserved if the vortices are not too much deadened. When old they are more and more sensible to accidental fluctuations, while the equidistance, in particular, is less and less well defined. The arrangement of vortices is indicated in the following figure (fig. 1).

FIG. 1.



An alternate periodic arrangement of vortices of this type has been observed many times. In the following diagram (fig. 2), taken from a paper by Osborne Reynolds (45), a series of spiral-shaped eddies is shown which bears some resemblance to the above arrangement. The diagram indicates the way in which a thin stream of liquid becomes unstable when moving through another liquid.

FIG. 2.



A somewhat similar arrangement of vortices is produced by the blades of a screw propeller (46) and a vibration of the vortex field has been noticed in some experiments on the flow of water round a model balloon (47), while Borne (48) has recently verified the fact that vortex filaments are formed alternately in the flow of air round different obstacles and has obtained some photographic records of the phenomena in question.

Experiments analogous to those of Bénard have also been made by Rubach (49), Kármán (49) and others with similar results. In the case of a circular cylinder moving through a liquid Rubach (50) found that two vortices with opposite directions of rotation are soon formed behind the cylinder and their strength continually increases, new rotating fluid being derived from the "layer of separation" which first forms behind the cylinder. The pair of vortices recedes from the cylinder with a velocity which is small compared with the progressive velocity of the cylinder and its pair of vortices relative to the stationary fluid. This state of affairs, however, is unstable; a periodic motion soon sets in with a continual formation of new vortices from opposite sides of the cylinder.

A mathematical theory of the two rows of vortices considered above has been given by Kármán (49), who finds that under certain circumstances such a system is stable.

In the case of a perfect incompressible fluid, the motion of a system of isolated rectilinear vortex filaments whose axes are all parallel, may be studied by a well-known method (51). Since each vortex moves with the fluid, the velocity of a vortex, A_p, can be calculated from the stream-line function due to the remaining vortices. Writing $z = x + iy$, $w = x - iy$, the equations of motion of the vortex A_p are contained in the single equation

$$\frac{dz_p}{dt} = \frac{i}{2\pi} \sum_q{}' \frac{k_q}{w_p - w_q},$$

where k_q is the strength of the vortex A_q and the prime denotes that in the summation q does not take the value p.

To study the small vibrations of the system, we write $z_p + \zeta_p$ instead of z_p, $w_p + \xi_p$ instead of w_p and neglect terms of order higher than the first in the small quantities ζ_p, ξ_p. We thus obtain

$$\frac{d\zeta_p}{dt} = -\frac{i}{2\pi} \sum_q{}' k_q \frac{\xi_p - \xi_q}{(w_p - w_q)^2}.$$

So far the work is quite general. Now let us assume that the vortices in the first row are all of strength k and that their undisturbed positions are given by

$$z_q = ql, \quad q = 0, \pm 1, \pm 2, \cdot \cdot \cdot$$

Let us assume, moreover, that the vortices of the second row are each of strength $-k$, and that their undisturbed positions are given by

$$z_r = (r+\tfrac{1}{2})l + ih,\ r = 0,\ \pm 1, \pm 2, \cdot\cdot\cdot$$

For simplicity each row of vortices is supposed to extend to infinity both ways.

Using the inferiors p, q, for vortices in the first row, and r, s, for vortices in the second row, we obtain the equations

$$\frac{d\zeta_p}{dt} = -\frac{ik}{2\pi}\sum_{q=-\infty}^{\infty}{}' \frac{\xi_p - \xi_q}{(p-q)^2 l^2} + \frac{ik}{2\pi}\sum_{r=-\infty}^{\infty} \frac{\xi_p - \xi_r}{[(p-r-\frac{1}{2})l - ih]^2};$$

$$\frac{d\zeta_s}{dt} = +\frac{ik}{2\pi}\sum_{r=-\infty}^{\infty}{}' \frac{\xi_s - \xi_r}{(s-r)^2 l^2} - \frac{ik}{2\pi}\sum_{q=-\infty}^{\infty} \frac{\xi_s - \xi_q}{[(s-q+\frac{1}{2})l + ih]^2}.$$

Now consider the disturbance in which ζ_p, ζ_s, ξ_p, ξ_s, are the unambiguous parts of the expressions $\zeta_0 e^{\pm ip\phi}$, $\zeta_1 e^{\pm i(s+\frac{1}{2})\phi}$, $\xi_0 e^{\pm ip\phi}$, $\xi_1 e^{\pm i(s+\frac{1}{2})\phi}$, respectively, ϕ being a real quantity independent of p and s and unambiguous, while ζ_0, ζ_1, ξ_0, ξ_1 are real or complex quantities which may involve the ambiguity $\pm$. Let us assume, moreover, that ξ_0, ζ_0, ξ_1 and ζ_1 depend on t through a factor type $e\theta_t$. We then have

$$\zeta_0\theta + \lambda\xi_0 + i\mu\xi_1 = 0, \quad \zeta_1\theta - \lambda\xi_1 - i\nu\xi_0 = 0,$$

where

$$\lambda = \frac{ik}{\pi}\sum_{n=l}^{\infty} \frac{1-\cos n\phi}{n^2 l^2} - \frac{ik}{2\pi}\sum_{m=-\infty}^{\infty} \frac{1}{[(m-\frac{1}{2})l - ih]^2},$$

$$\mu = \frac{k}{2\pi}\sum_{m=-\infty}^{\infty} \frac{e^{\pm(m+\frac{1}{2})l\phi}}{[(m+\frac{1}{2})l + ih]^2},$$

$$\nu = \frac{k}{2\pi}\sum_{m=-\infty}^{\infty} \frac{e^{\pm(m+\frac{1}{2})i\phi}}{[(m+\frac{1}{2})l + ih]^2}.$$

In a similar way we find that

$$\xi_0\theta - \lambda\zeta_0 + i\nu\xi_1 = 0, \quad \xi_1\theta + \lambda\zeta_1 + i\mu\zeta_0 = 0.$$

Eliminating ξ_0, ξ_1, ζ_0, ζ_1, we obtain the equation

$$(\theta^2 + \lambda^2 + \mu^2)(\theta^2 + \lambda^2 + \nu^2) - \lambda^2(\mu - \nu)^2 = 0$$

which is satisfied by

$$\theta = \pm\tfrac{i}{2}(\mu - \nu) \pm i\sqrt{\lambda^2 + \tfrac{1}{4}(\mu + \nu)^2}.$$

For stability it is necessary that θ should be a purely imaginary quantity. Now $\mu - \nu$ and $\mu + \nu$ are real, consequently for stability $\lambda^2 + \frac{1}{4}(\mu+\nu)^2$ must be positive for all real values of the quantity ϕ, which specifies the relations between the phases of the different vortices. Now when $\phi = \pi$, $\mu + \nu = 0$, hence for stability λ must also vanish when $\phi = \pi$, otherwise λ^2 would be negative and there would be two values of θ with a positive real part. The equation $\lambda^2 = 0$ reduces, when $\phi = \pi$, to $\cosh^2\frac{h\pi}{l} = 2$, or

$$\frac{h}{l} = 0.283 \ \ldots \qquad (1)$$

For other ratios of h to l the system of vortices is unstable. A complete proof that the system is stable for all displacements when h and l are connected by the relation (1) has not been given, and in spite of Kármán's assertion, there is some doubt about the truth of the theorem, judging from the report of a paper presented to the Royal Society of Edinburgh on March 1, 1915, by H. Levy. It is easy to see, however, that $\lambda^2 + \frac{1}{4}(\mu+\nu)^2$ is positive when $\phi = 0$.

The velocity U with which the whole system of vortices moves is given by the formula

$$U = \frac{k}{\pi}\sum_{m=0}^{\infty} \frac{h}{(m+\frac{1}{2})^2 l^2 + h^2} = \frac{k}{2l}\tanh\frac{\pi h}{l} = \frac{k}{l\sqrt{8}}$$

Kármán has used the two rows of vortices to obtain a theory of resistance in which the resistance encountered by a body moving with uniform velocity, V, in a perfect fluid is expressed in terms of V, h, l, and the width of the body in a direction perpendicular to the direction of motion. The quantity l may be obtained experimentally from observations of the periodic system of vortices in the wake behind the body.

In connection with the flow produced by a moving right circular cylinder, L. Föppl (50) has investigated whether there are any places behind a cylinder in uniform motion, where two equal vortices with opposite senses of rotation can be placed so as to be at rest relative to the cylinder. By considering the images of the vortices in the cylinder, he finds that the vortices must lie in symmetrical positions on the curves $\pm 2y = r - \frac{1}{r}$, where r is the distance of a point from the axis of the cylinder, the radius of the cylinder being unity. This result has been tested by an examination of Rubach's photographs and agrees very well with the measurements. The strength of the vortices is greater the farther they are from the axis of the cylinder.

3. Let us now see what relation some of the preceding results and theories may have to atmospheric problems.

In his memoir "Über atmosphärische Bewegungen" (52) Helmholtz has made some remarks on the origin of depressions and anticyclones and has considered the possibility of surfaces of discontinuity in the atmosphere, these being surfaces which separate masses of air with different velocities and different temperatures or densities. Such surfaces are sooner or later broken up, eddies are formed, and the masses of air at different temperatures intermingle. Helmholtz thus regards instability as a more powerful cause than friction in establishing a transition stratum within which the change of density takes place gradually. The effect of viscosity in smoothing out discontinuities may be studied by considering some of the well-known problems in the theory of the conduction of heat, wherein a discontinuity in the initial conditions instantly disappears after a time. Helmholtz remarks (53) that—

> As in the neighborhood of the Equator the air of the earth's surface is warmed and rises, so in the neighborhood of the poles it is cooled and sinks. The cold layers will endeavor to flow separately to the earth and form east winds; above them the vacant place must be filled and the warm air blows there as a west wind or cyclone. It would be possible for there to be equilibrium if the lower cold layers did not require a more rapid motion of rotation owing to friction. The spreading out of the polar east winds, if it is indeed recognizable in its principal features, takes place very irregularly, since the cold pole does not coincide with the rotation-pole of the earth and low hills have considerable influence. Through such irregularities it happens that the anticyclonic movement of the lower layers and the gradually increasing cyclone of the upper layers, which is to be expected at the pole, resolve themselves into a large number of irregularly moving cyclones and anticyclones with a preponderance of the former.

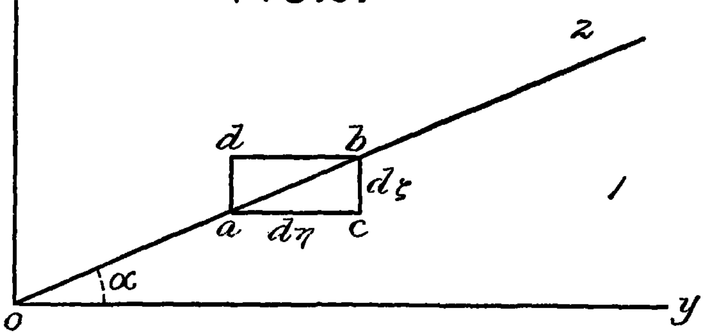
This idea has been recently taken up and developed by F. M. Exner (54), who has combined it with some results obtained by M. Margules (55). The fundamental hypotheses may be briefly stated as follows:

> Let us suppose, with Helmholtz, that in the neighborhood of the pole the cooled air sinks and endeavors to spread out toward the south beneath the overlying warmer layers, like a heavy fluid beneath a lighter one. By the deviating force of the earth's rotation the flow of cold air toward the south should be directed toward the west, the cold air should therefore enter the high latitudes as east winds; south of this the warm west wind should prevail. According to the calculations of Helmholtz and Margules, the cold layer can lie in a state of equilibrium like a wedge below the warm air. On account of the friction at the earth's surface which the lowest layers experience, the cold east wind and the warm west wind are retarded; the surface of separation is consequently not stable but bends toward the horizon. Cold air flows southward, while warm air flows northward.

This is an overturning of the layers in the sense of Margules, whereby kinetic energy is set free by the work of gravity. Exner has made some calculations to ascertain whether, under plausible assumptions as to the magnitude of the friction, this is sufficient to account for the great air movements, and he finds that this is the case. He then says:

> Since the friction on a parallel of latitude is very different for different lengths, there is a very different production of kinetic energy in different places. This signifies the generation of depressions at certain parts of a parallel of latitude, which are characterized by particularly great hindrances to the east-west air motion. The growth of depressions may consequently be connected with certain spots on the earth's surface. Among these the continent of Greenland plays a particularly important part, for the cold east winds are dammed at its east front and thrown toward the south. On account of the lack of observations in high latitudes this conclusion has unfortunately not been sufficiently confirmed.

FIG. 3.



An examination of the weather maps of the Northern Hemisphere (56) will show that this theory of Helmholtz and Exner does account for the general features of the pressure distribution, as there is frequently a circle of lows at about the same latitude as Greenland, and this ring of lows is surrounded by a belt of highs. The arrangement of these lows and highs bears some resemblance to Bénard's two rows of vortices, but unfortunately the lows are more numerous than the highs, so that a high is not always equidistant from two consecutive lows as in Bénard's arrangement.

Very little has been done in the theory of the stability of a large number of isolated vortices, that might conceivably have an application to atmospheric problems. Perhaps the arrangement considered by Lord Kelvin in his paper "On the stability and small oscillations of a perfect liquid full of nearly straight coreless vortices" (57) might with advantage be transferred to the surface of a sphere and studied more fully. More progress has been made in the theory by using the idea of a surface of discontinuity, although, as Exner remarks, there is no direct evidence that a sharp discontinuity in temperature occurs. Nevertheless the results which are obtained by using the idea may be expected to closely resemble the actual conditions.

When the surface of discontinuity is stationary, its inclination may be deduced from a formula given by Margules (58).

Using the equations for stationary rectilinear motion in the form

$$\frac{1}{\rho_1}\frac{\partial p_1}{\partial y}=2\omega\sin\phi.u_1, \qquad \frac{1}{\rho_1}\frac{\partial p_1}{\partial z}=-g,$$

$$\frac{1}{\rho_2}\frac{\partial p_2}{\partial y}=2\omega\sin\phi.u_2, \qquad \frac{1}{\rho_2}\frac{\partial p_2}{\partial z}=-g,$$

where u is the velocity parallel to the axis of X, p the pressure, ρ the density of the air, ω the angular velocity of the Earth, and ϕ the latitude, he writes

$$p_c-p_a=\frac{\partial p_1}{\partial y}.d\eta=2\omega\rho_1u_1\sin\phi.d\eta.$$

$$p_b-p_c=\frac{\partial p_1}{\partial z}.d\zeta=-g\rho_1.d\zeta.$$

$$p_b-p_d=\frac{\partial p_2}{\partial y}.d\eta=2\omega\rho_2u_2\sin\phi.d\eta.$$

$$p_d-p_a=\frac{\partial p_2}{\partial z}.d\zeta=-g\rho_2.d\zeta.$$

therefore

$$2\omega\rho_1u_1\sin\phi.d\eta-g\rho_1d\zeta=p_b-p_a=2\omega\rho_2u_2\sin\phi.d\eta-g\rho_2.d\zeta,$$

and so

$$\tan\alpha=\frac{d\zeta}{d\eta}=\frac{2\omega\sin\phi}{g}.\frac{\rho_1u_1-\rho_2u_2}{\rho_1-\rho_2}.$$

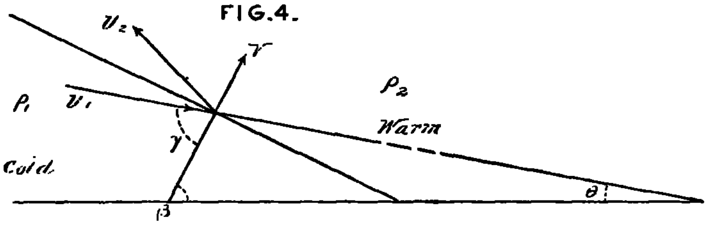
Margules also obtains a more exact formula by taking into account the curvature of the Earth. Sandström (59) has shown further that it is possible to derive the variation in intensity of a vortex sheet from the inclination of the surface of discontinuity as well as from the temperature and distribution of humidity in the neighborhood of the surface. He also gives a criterion depending on the relative velocity (60). Some evidence of the existence of a condition in the atmosphere closely resembling a surface of discontinuity has been obtained by W. Schmidt in his observations of air-waves in valleys (61). He finds that the amplitudes of the waves increase continuously as the warm föhn current aloft sinks to the surface, until they suddenly end when the föhn breaks through. He considers that the air-waves are formed at the upper surface of the cold valley wind over which blows the warm current of the föhn. The appearance of the air-waves is frequently associated with an advancing depression, and may thus be regarded as a weather prognostic.

The motion of surfaces of discontinuity in the atmosphere may perhaps be studied with the aid of some well-known theorems relating to the propagation of waves of discontinuity (62). One of these theorems may be deduced at once from the equation of continuity. It states that if V is the velocity of the surface of discontinuity in a direction at right angles to itself, v_1 and v_2 the component velocities of the air on the two sides of the surface in a direction at right angles to the surface, ρ_1 and ρ_2 the densities of the two contiguous masses of air, then

$$\rho_1(V-v_1)=\rho_2(V-v_2) \quad .$$

Let us apply this equation to the case of a line squall on the supposition that the inclination of the surface of discontinuity to the horizon is as shown in figure 4.

FIG. 4.



Let U_1, U_2, be the velocities of the air on the two sides of the surface of discontinuity.

Since the rate of advance of the line of the squall is a little greater than the surface velocity of the colder air (63) we have

$$V \cos\beta > U_1 \cos\theta$$

Now $v_1 = U_1 \cos \gamma$, and $\cos\theta = \cos\beta\cos\gamma + \sin\beta\sin\gamma$, therefore $\cos \theta > \cos\gamma\cos\beta$, and so $v_1 < U_1\cos\theta\sec\beta < V$. It follows then that V is also greater than v_2. This means that warm air flows across the surface of discontinuity and mixes with the cold air, a result which may perhaps be regarded as an illustration of the principle that heat always flows freely from the warmer mass to the cold and not vice versa. The supposition made with regard to the inclination of the surface of discontinuity and its velocity, is thus consistent with the above equation of continuity.

The flow of fluid past a spherical obstacle has an interesting application to the atmospheric problem of the flow of air past a hemispherical mountain, as has been pointed out by W. Schmidt (64). The influence of the compressibility of the air has been considered by Y. Okada (65) and has been found to be negligible, provided the velocity of the air is small compared with the velocity of sound. When the surface of the mountain is treated as a half cylinder, the investigations in the paper of L. Föppl (50) become of interest. By reducing the cylinder to rest, we see that it is possible for a stationary vortex to form behind the mountain, a result which agrees with observations. L. Föppl's investigation of the stability of the two vortices behind a circular cylinder is thus of interest for the atmospheric problem. It must be remembered that in this case the two vortices must always be images of one another in the plane of symmetry, i. e., the plane which divides the cylinder into two halves; the displacements of the two vortices are consequently symmetrical with regard to this plane and the arrangement is stable, whereas for asymmetrical displacements the arrangement is unstable.

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III. THE DISTRIBUTION OF THE RAINFALL IN THE WESTERN UNITED STATES.

By B. C. WALLIS, B. Sc. (Economics), F. R. G. S., F. S. S.

[Dated: North Finchley, England, Feb. 24, 1915.]

In this REVIEW for January, 1915, the writer mapped in some detail and discussed the distribution of rainfall intensity in the eastern United States; the present paper is a similar discussion of the rainfall intensity in the western portion of the Republic.

The accompanying 12 monthly maps of equipluves (figs. 31–42) present a notable regularity almost throughout the year, a very wet area gradually fades off into a very dry district. The exceptional month is October, when the raininess is uniformly below the average, and the elevated lands are wetter than the lowlands. The second general feature is the absence of very marked raininess or dryness on the mountains at any time of the year. This fact is well shown by the graphs for the mountain divisions (fig. —). Consequently, in a broad way, the West contains three regions with three types of rainfall: (1) The Far West, including the coast lands, with great rainfall intensity throughout the period November to March, i. e., *winter rains;* (2) the Mountains, never very wet, never very dry; (3) the Eastern Slopes, with great rainfall intensity in the north from April to June, and in the south from July to September, i. e., *summer rains.*

In January the equipluves run north and south and raininess decreases steadily eastwards. This month marks the climax of the influences which cause rain and which are due, in the main, to the winds from the Pacific Ocean.

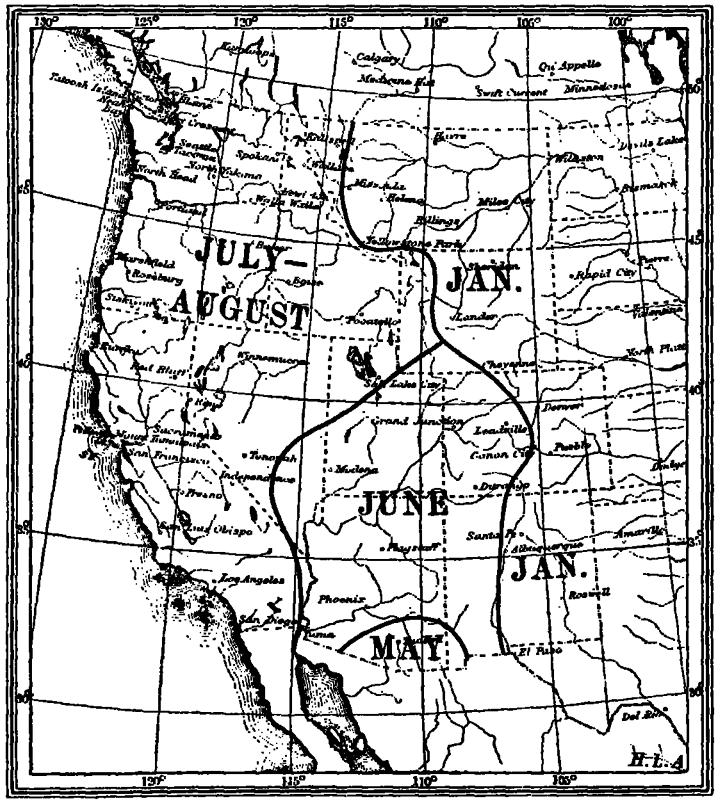

FIG. 24.—Map showing the driest months in the western United States.

In February the rainfall influences begin to weaken along the northwest coast and raininess increases on the eastern slopes.

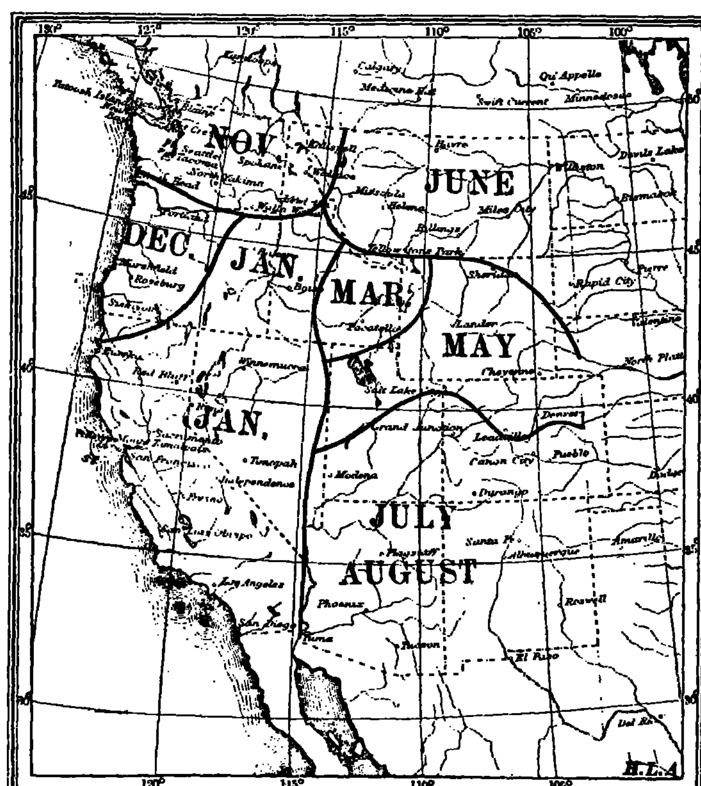

FIG. 25.—Map showing the wettest months in the western United States.